
# International Conference on Optimization, Simulation and Control

# On a Problem of Optimal Control of the Vibration of a Rod Consisting of Two Non-Homogenous Segments with a Restriction at an Intermediate Time

Barseghyan V.[1,2]; Solodusha S.[3]

[1]Institute of Mechanics NAS RA, Yerevan, Armenia, [2]Yerevan State University, Yerevan, Armenia , [3]Melentiev Energy Systems Institute SB RAS

## Abstract

We consider an optimal boundary control problem for a one-dimensional wave equation consisting of two non-homogenous segments with piecewise constant characteristics. The wave equation describes the longitudinal vibrations of an non-homogeneous rod or the transverse vibrations of an non-homogeneous string with given initial, intermediate, and final conditions. We assume that wave travel time for each of the sections is the same. The control is carried out by shifting one end with the other end fixed. The quality criterion is set on the entire time interval. A constructive approach to building an optimal boundary control is proposed. The results obtained are illustrated by an analytical example.

## Introduction

Many researchers pay attention to the study of control problems and optimal control problems for vibration processes [1-10]. Modeling and control of dynamic systems with intermediate conditions is an actively developed direction in modern control theory. In particular, works [2-10] address the study of such problems. The control problems of inhomogeneous elastic vibrations are not studied enough. The study of problems for such heterogeneous distributed systems is devoted, in particular, to the works [8-10].

This work aims to develop a constructive approach to building an optimal boundary control function for an inhomogeneous wave equation consisting of two heterogeneous sections with given initial, intermediate, and final conditions with a quality criterion given over the entire time interval.

## Problem statement

We consider longitudinal vibrations of a piecewise homogeneous rod located along the segment $-l_1 \le x \le l$ and comprising two subsegments.

The segment $-l_1 \le x \le 0$ has a linear density $\rho_1 = const$, Young's modulus $k_1 = const$ and a wave velocity $a_1 = \sqrt{\frac{k_1}{\rho_1}}$. The second segment $0 \le x \le l$ has a linear density $\rho_2 = const$, Young's modulus $k_2 = const$ and a wave velocity $a_2 = \sqrt{\frac{k_2}{\rho_2}}$. As in [9], we assumed that the lengths $l_1$ and $l$ of the rod segments are such that the wave velocity on $-l_1 \le x \le 0$ coincides with the wave velocity on $0 \le x \le l$ , i.e.

$$\frac{l_1}{a_1} = \frac{l}{a_2}. \quad (1)$$

Let the state (longitudinal vibrations) of the rod (or transverse vibrations of the string) be described by the function $Q(x,t)$, $-l_1 \le x \le l$ , $0 \le t \le T$, and the deviations from the equilibrium state satisfy the following wave equation

$$\frac{\partial^2 Q(x,t)}{\partial t^2} = \begin{cases} a_1^2 \frac{\partial^2 Q(x,t)}{\partial x^2}, -l_1 \le x \le 0, 0 \le t \le T, \\ a_2^2 \frac{\partial^2 Q(x,t)}{\partial x^2}, 0 \le x \le l, 0 \le t \le T, \end{cases} \quad (2)$$

## Problem statement

with the boundary conditions

$$Q(-l_1,t) = \mu(t),\ Q(l,t) = 0,\ 0 \le t \le T, \quad (3)$$

and with the conjugation conditions at the connection point $x = 0$ of the segments

$$Q(0-0,t) = Q(0+0,t),\ a_1^2 \rho_1 \frac{\partial Q(x,t)}{\partial x}\Big|_{x=0-0} = a_2^2 \rho_2 \frac{\partial Q(x,t)}{\partial x}\Big|_{x=0+0}. \quad (4)$$

Let there be given initial (for $t = t_0 = 0$) and final (for $t = T$) conditions

$$Q(x,0) = \varphi_0(x), \frac{\partial Q(x,t)}{\partial x}\Big|_{t=0} = \psi_0(x), \quad -l_1 \le x \le l, \quad (5)$$

$$Q(x,T) = \varphi_T(x), \frac{\partial Q(x,t)}{\partial x}\Big|_{t=T} = \psi_T(x), \quad -l_1 \le x \le l. \quad (6)$$

Additionally, let there be given at some intermediate moment of time $t_1$ $(0 = t_0 < t_1 < t_2 = T)$ an intermediate state in the form:

$$Q(x,t_1) = \varphi_1(x), \quad -l_1 \le x \le l. \quad (7)$$

In (3), functions $\mu(t)$ is control actions (boundary control).

It is assumed that $Q(x,t) \in C^2(\Omega_T)$, where

$$\Omega_T = \{(x,t): x \in [-l_1, l], t \in [0,T]\}$$

and $\varphi_i(x) \in C^2[-l_1, l], i = \overline{0,2}$, $\varphi_0(x), \varphi_T(x) \in C^1[-l_1, l]$.

We also assume that for all functions the following consistency conditions are Satisfied

$$\mu(0) = \varphi_0(-l_1), \dot{\mu}(0) = \psi_0(-l_1),\ \varphi_0(l) = \psi_0(l) = 0,$$
$$\mu(t_1) = \varphi_1(-l_1), \varphi_1(l) = 0, \mu(T) = \varphi_T(-l_1), \quad (8)$$
$$\dot{\mu}(T) = \psi_T(-l_1), \varphi_T(l) = \psi_T(l) = 0.$$

Let us formulate the following problem of optimal boundary control of oscillations for system (2) with given values at intermediate times.

Among the possible controls $\mu(t)$, $0 \le t \le T$, (3) it is required to find such an optimal control that provides transition of the oscillatory motion of system (2) from a given initial state (5) to the final state (6), (7) at the same time ensuring the fulfillment of condition (1.7) and minimizing the functional

$$\int_0^T \mu^2(t)\, dt. \quad (9)$$

## Results

Since the functional (9) is the square of the norm of a linear normed space, the problemof determining the optimal control for each$k = 1, 2, \dots$ can be considered as aproblem of moments [1]. Therefore, the solution can be constructed usingthe algorithm for solving the problem of moments.

We will receive

$$\mu_1^0(\tau) = \begin{cases} \mu_1^{(1)0}(\tau) = \frac{1}{(\rho_1^0)^2}[p_1^0 \sin\lambda_1(T-\tau) + q_1^0 \cos\lambda_1(T-\tau) + \gamma_1^0 \sin\lambda_1(t_1-\tau)], 0 \le \tau \le t_1, \\ \mu_1^{(2)0}(\tau) = \frac{1}{(\rho_1^0)^2}[p_1^0 \sin\lambda_1(T-\tau) + q_1^0 \cos\lambda_1(T-\tau)], t_1 < \tau \le t_2 = T \end{cases}$$

## Results

Where

$$(\rho_1^0)^2 = \frac{T}{2}\left((q_1^0)^2 + (p_1^0)^2\right) + \frac{t_1}{2}\left((\gamma_1^0)^2 + 2\gamma_1^0(p_1^0 \cos\lambda_1 (T-t_1) - q_1^0 \sin\lambda_1 (T-t_1))\right) +$$
$$+\frac{1}{\lambda_1}\Bigg(p_1^0 q_1^0 \sin^2 \lambda_1 T - \frac{(\gamma_1^0)^2}{2} \sin\lambda_1 t_1 \cos\lambda_1 t_1 + \gamma_1^0(q_1^0 \sin\lambda_1 T - p_1^0 \cos\lambda_1 T)\sin\lambda_1 t_1 +$$
$$+\frac{(q_1^0)^2 - (p_1^0)^2}{2} \sin\lambda_1 T \cos\lambda_1 T\Bigg),$$

$$p_1^0 = \frac{1}{2\Delta}\left((f_{11}^2 - e_{11}g_{11})C_{11}(T) + (b_{11}g_{11} - c_{11}f_{11})C_{21}(T) + (e_{11}c_{11} - b_{11}f_{11})C_{11}(t_1)\right),$$
$$q_1^0 = \frac{1}{2\Delta}\left((b_{11}g_{11} - c_{11}f_{11})C_{11}(T) + (c_{11}^2 - a_{11}g_{11})C_{21}(T) + (a_{11}f_{11} - b_{11}c_{11})C_{11}(t_1)\right),$$
$$\gamma_1^0 = \frac{1}{2\Delta}\left((e_{11}c_{11} - b_{11}f_{11})C_{11}(T) + (a_{11}f_{11} - b_{11}c_{11})C_{21}(T) + (b_{11}^2 - a_{11}e_{11})C_{11}(t_1)\right),$$

$$\Delta = \frac{1}{2}[(f_{11}^2 - e_{11}g_{11})C_{11}^2(T) + (c_{11}^2 - a_{11}g_{11})C_{21}^2(T) + (b_{11}^2 - a_{11}e_{11})C_{11}^2(t_1)] +$$
$$+(e_{11}c_{11} - b_{11}f_{11})C_{11}(t_1)C_{11}(T) + (b_{11}g_{11} - c_{11}f_{11})C_{21}(T)C_{11}(T) + (a_{11}f_{11} -$$
$$b_{11}c_{11})C_{11}(t_1)C_{21}(T).$$

Applying the approach proposed above, we constructed an optimal boundary control for $n = 1$ $(k = 1)$ and the corresponding string deflection function:

for $0 \le t \le t_1$

$$Q_1^0(x,t) = \begin{cases} V_1^0(t)\ \sin\frac{\pi}{l_1}x + \frac{1}{2}\left(1 - \frac{x}{l_1}\right)\mu_1^{1(0)}(t), -l_1 \le x \le 0, \\ V_1^0(t)\ \sin\frac{\pi}{l}x + \frac{1}{2}\left(1 - \frac{x}{l}\right)\mu_1^{1(0)}(t), 0 \le x \le l, \end{cases}$$

for $t_1 < t \le t_2 = T$

$$Q_1^0(x,t) = \begin{cases} V_1^0(t)\ \sin\frac{\pi}{l_1}x + \frac{1}{2}\left(1 - \frac{x}{l_1}\right)\mu_1^{2(0)}(t), -l_1 \le x \le 0, \\ V_1^0(t)\ \sin\frac{\pi}{l}x + \frac{1}{2}\left(1 - \frac{x}{l}\right)\mu_1^{2(0)}(t), 0 \le x \le l. \end{cases}$$

## Conclusions

In this paper, we considered the problem of optimal boundary control of a one-dimensional wave equation describing transverse vibrations of a piecewise homogeneous string or longitudinal vibrations of a piecewise homogeneous rod. A constructive approach is proposed for building a optimal boundary control function for one-dimensional non-homogeneous oscillatory processes. In this case, the explicit expression of the optimal boundary control function is represented through the given initial and final functions of the deflection and velocities of the points of the distributed system. The results can be used when designing the optimal boundary control of non-homogeneous oscillation processes in physical and technological systems.

## Contact

Vanya Rafaelovich Barseghyan
Institute of Mechanics, National Academy of Sciences of Armenia, Yerevan, Armenia
Yerevan State University, Yerevan, Armenia
barseghyan@sci.am, barsegh@ysu.am
(37410) 523640

## References

1. Butkovskii, A.G.: Control Methods for Systems with Distributed Parameters. Nauka, Moscow (1975). (in Russin)
2. Barseghyan, V.R.: The Control Problem for Stepwise Changing Linear Systems of Loaded Differential Equations with Unseparated Multipoint Intermediate Conditions. Mechanics of Solids 53(6), 615–622 (2018).
3. Barseghyan, V.R.: The problem of optimal control of string vibrations. Intern. Appl. Mech. 56(4), 471–480 (2020).
4. Barseghyan, V.R.: The problem of optimal control of vibrations of a string with non-separated conditions into state functions at given intermediate time instants. Autom. Remote Control 81(2), 226–235 (2020).
5. Barseghyan, V., Solodusha, S.: Optimal Boundary Control of String Vibrations with Given Shape of Deflection at a Certain Moment of Time. Lecture Notes in Control and Information Sciences 12755, 299–313 (2021).
6. Barseghyan, V., Solodusha, S.: On One Problem in Optimal Boundary Control for String Vibrations with a Given Velocity of Points at an Intermediate Moment of Time. In: IEEE Proc. of International Russian Automation Conference (RusAutoCon), 343–349 (2021).
7. Barseghyan, V.R.: On the controllability and observability of linear dynamic systems with variable structure. In: IEEE Proc. of International Conf. Stability and Oscillations of Nonlinear Control Systems (Pyatnitskiy's Conference), 1–3 (2016).
8. Barseghian, V.R.: String vibration observation problem. In: 1-st International Conference. In: Control of Oscillations and Chaos Proceedings (Cat. No. 97TH8329) 2, 309–310 (1997). https://doi.org/10.1109/COC.1997.631351
9. Il'in, V.A.: Optimization of the boundary control of the vibrations of a rod consisting of two dissimilar parts. Dokl. Math. 84(2), 629–633 (2011).
10. Il'in, V.A.: On the bringing of the oscillations of an initially quiescent rod consisting of two different parts to an arbitrarily given state. Dokl. Math. 82(3), 955–958 (2010).