Stochastic Optimal Control Problem of Consumption and Pension Insurance Purchase with Uncertain lifetime

Batsukh.Ts, PhD[1]; Tserenbat..O, PhD[2] ; Gereltuya.T, MD[3]; Nyamsuren.D, PhD[3]
[1]University of Finance and Economics, [2]National University of Mongolia , [3]Mongolian University of Science and Technology

Abstract

In this paper we consider the optimal consumption and pension insurance premium with uncertain lifetime. This model is the pension insurance model, that is special case of the life insurance. By using the dynamic programming technique, we derive Hamilton-Jacobi-Bellman (HJB) equations for our model and obtain explicit solution for CARA utility function.
In the simulation of this model, we use a third-order B-spline function to construct the logarithm for force of mortality of living time.
Evaluation of B-spline parameters estimated by maximum likelihood estimation tested with criteria of the modified chi-squared goodness of the fit statistic.
A simulation has been carried out showing how policy decisions change based on different values of aversion risk parameters of bequest and utility functions.

Introduction

Beginning in the 1960's, many researchers constructed models to analyze for life insurance and the behavior of investment for an individual under an uncertain lifetime.
Yaari (1965) is a starting point for the modern research on the demand for life insurance. Yaari considered the problem of life insurance under an uncertain lifetime for an individual.
Researchers employed the same dynamic programming technique to attack this problem and to obtain explicit solutions for Constant Relative Risk Aversion (CRRA) case.
Setting up the force of mortality and the living time distribution are crucially important in actuarial science, health, life insurance. Several contributions have been made in force of mortality, living time distribution.
Tserenbat Oirov (2021) evaluated the logarithm of the force of mortality in the form of the third order B-spline, where its parameters have been estimated by the Maximum Likelihood Estimation.
In the present work we employed risk-free asset in the market and stochastic wage. We applied survival analysis with B-spline estimation for the optimal solution of the problem.

Pension insurance model

Stochastic optimal control problem directed for defining the optimal consumption and optimal premium rate leads to the maximizing expectation of the consumer's utility of the uncertain time and written as

$$\max_{\{c(t),P(t)\}} \left\{ \int_0^{\bar{T}} \left[f(t)\Phi\big(t,Z(t)\big) + S(t)\exp\left\{-\int_0^t \rho(\tau)d(\tau)\right\} U\big(c(t)\big)dt \right] \right\}$$

subject to

$$dW(t) = \big[r(t)W(t) + \mu_y(t) - c(t) - \theta(t)P(t)\big]dt + \sigma_y(t)dB(t)$$
$$Z(t) = W(t) + P(t), \qquad W(\tilde{\tau}) = W\_\{\bar{\tau}\}$$

Here, $Z(t)$ is the contingent bequest.

Explicity solution's necessary condition

Let's choose the following forms of utility function of consumption and bequest:

$$\begin{cases} u(t,c9t)) = -\frac{\alpha}{\beta}\exp(-\beta\, c(t)) + \gamma \\ \Phi\big(t,Z(t)\big) = -\frac{A(t)}{k}\exp(-kZ) + \nu \end{cases}$$

In this case, we get a value function $F(t,W) = -\frac{b(t)}{a(t)}\exp\{-a(t)W\}$.

Let's define the optimal consumption and premium as function of $u(t,,c(t))\ and\ \ \varphi(t,,Z)$. We have

$$c^*(t) = \frac{1}{\beta}\left(a(t)W - \ln\left(\frac{b(t)}{\alpha}\right)\right)$$

and

$$\begin{cases} A(t) = \theta(t)b(t) \\ P(t) = \left(\frac{a(t)}{k} - 1\right)W \end{cases}$$

Hence we can find the Berrnoulli equation for $a(t)$

$$a'(t) + \big(r(t) + \theta(t)\big)a(t) = \left(\frac{1}{\beta} + \frac{\theta(t)}{k}\right)a^2(t)$$

and for $b(t)$ $\quad [Ln\ b(t)]' - \frac{a(t)}{\beta}\ln b(t) = D(t)$

Here, $D(t) = \rho^*(t) - r(t) - \theta(t) + \mu_y(t)a(t) - \frac{a(t)}{\beta}\ln\alpha - \frac{1}{2}\sigma_y^2(t)a^2(t)$

Simulation

- Study of an age of Mongolia population dynamics

In this simulation we consider Mongolian population data of 2019.
For the estimation, we used a third-order B-spline function to construct the logarithm for force of mortality of living time. Using the estimation of force of mortality, survival and density functions were found.

- Wage and wealth dynamics

The interest rate function is estimated by linear regression model based on Mongolian the weighted average of the interest rate data for 2008-2021. The wage trend and standard error functions are estimated by linear regression model based on the national average real wage data for $r_\tau = 0.1 - 0.002\,\tau$, $\mathrm{Y}_\tau = 0.22 + 0.11\tau, \sigma = 0.21$.

- Discount, utility and bequest functions

Let's choose $\rho^* = 0.15$, then discount coefficient becomes $\rho(t) = 0.15 - \lambda(t)$. If we suppose that insurance customer begin to purchase at age 27, then an insurance premium rate can be written to be $\theta(t) = \lambda(t) + 0.9/\tau$. If we choose α, β to be 2 and 3 each, then utility function for consumer is $U(c) = -\frac{2}{3}\exp(-3\,c) + 5$.

- Optimal control functions and their dynamic

Using model's parameters, we can calculate consumer's and bequest utility function parameters. By using these parameters, wealth, consumption, insurance purchase and insurance premium rate are estimated.

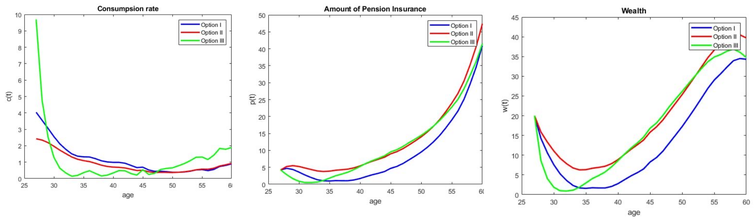

Figure 3. Changes in consumption rate , Amount of Pension insurance purchases and Wealth functions for different parameter values.

Figure 1 The wage function.

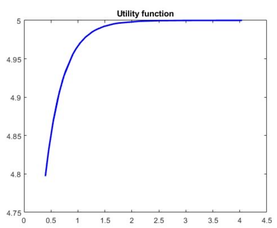

Figure 2. The utility function .

Table 1. Value of different parameters

	Parameter of bequest function-k	Parameter of utility function-β
Option I	0.5	3
Option II	0.5	5
Option III	1.2	3

Conclusions

In this paper, we set up a new model to investigate the problem of optimal pension insurance purchase, consumption for stochastic wage under an uncertain lifetime. An analytical solution of above portfolio optimization problem was investigated. Following results obtained by the estimation:

- Optimal consumption and optimal insurance premium are linear in wealth with time variable coefficient.
- When the aversion risk parameter of utility function increases, consumption level decreases and insurance purchase level increases
- When the aversion risk parameters values of the bequest function increases, the level of consumption initially decreases rapidly, but the purchase of insurance increases rapidly.

Contact

[Batsukh. Ts, Tserenbat. O, Gereltuya. T, Nyamsuren.D]
[[1]University of Finance and Economics, [2]National University of Mongolia , [3]Mongolian University of Science and Technology]
[Ulaanbaatar, Mongolia]

[batsukh.ts@ufe.edu.mn , O_tsbat@yahoo.com, Gereltuya.t@must.edu.mn, Donyam@must.edu.mn]

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