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Fractional maximization problem

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Abstract

In this talk, we consider a problem of maximizing a ratio of convex and concave functions defined positively on a convex set. We reduce the problem to a quasiconvex maximization problem and apply the global optimality conditions. Based on the global optimality conditions, we develop an algorithm for solving the problem. The algorithm uses an approximation of the level set of the objective function. Numerical results are given on examples of ratio of two quadratic functions.

Introduction

We consider fractional optimization of the following type:

$$\max_{x\in D}\left\{\varphi(x)=\frac{f(x)}{g(x)}\right\}, \tag{1}$$

where $D\subset\mathbb{R}^n$ is a subset, and $f(x)$ is convex, $g(x)$ is concave on D, $f(x)$ and $g(x)$ are positive on D. That problem is the fractional optimization problem. Problems (1) have many applications in economics and engineering. For example, problems such as minimization of average cost function [1] and minimizing the ratio between the amount of resource wasted a class of fractional programming.

Methods and Materials

Fractional Maximization and Global Optimality Conditions:

Introduce the level set of the function $\varphi(x)$ for a given $C>0$.

$$L(\varphi,C)=\{x\in D \mid \varphi(x)\le C\}.$$

Lemma 1. *The set $L(\varphi,C)$ is convex.*

Proof. Since $g(x)>0$ on D, then $\varphi(x)\le C,\ \forall x\in D$ can be written as follows:

$$f(x)-Cg(x)\le 0, \qquad \forall x\in D.$$

a set defined by

$$M=\{x\in D \mid f(x)-Cg(x)\le 0\}$$

is convex which implies convexity of $L(\varphi,C)$.

Definition 1. *A function $f: D\to\mathbb{R}$ is said to be quasiconvex on D if*

$$f(\alpha x+(1-\alpha)y)\le \max\{f(x),f(y)\}$$

hold for all $x,y\in D$ and $\alpha\in[0,1]$.

Lemma 2. *[4] The function $f(x)$ is quasiconvex on D if and only if the set $L(\varphi,C)$ is convex for all $C\in\mathbb{R}$.*

Then it is clear that the function $\varphi(x)$ is quasiconvex on D. The optimality conditions for a quasiconvex maximization problem were given in [2]. Applying this result to problem (1), we obtain the following proposition.

The optimally condition will be formulated as follows [2].

Theorem 1. *Let z be a solution to problem (1), and let $E_C(\varphi)=\{y\in\mathbb{R}^n \mid \varphi(y)=C\}$.*
Then

$$\langle\varphi'(y),x-y\rangle\le 0 \tag{2}$$

For all $y\in E_C(\varphi)$ and $x\in D$.

If in addition $\varphi'(y)\ne 0$ holds for all $y\in E_{\varphi(z)}(\varphi)$, then condition (2) is sufficient for $z\in D$ to be a global solution to problem (1).

Condition (2) can be simplified as:

$$\sum_{i=1}^{n}\left\{\frac{\partial f(y)}{\partial x_i}g(y)-\frac{\partial g(y)}{\partial x_i}f(y)\right\}\left(\frac{x_i-y_i}{g^2(y)}\right)\le 0$$

for all $y\in E_{\varphi(z)}(\varphi)$ and $x\in D$.

These global optimality conditions are based on the Global Search Theory developed by A.S. Strekalovsky [3].

Algorithm and Approximation Set

Definition 2. *The set A(z) defined for a given m by*

$$A_z^m=\left\{y^1,y^2,\dots,y^m \mid y^i\in E_{\varphi(z)}(\varphi)\cap D, i=1,2,\dots,m\right\}$$

is called an approximation set.

Lemma 3. *If there are a point $y^i\in A_z^m$ and a feasible point $z\in D$ such that*

$$\langle\varphi'(y^j),u^j-y^j\rangle>0$$

then $\varphi(u^j)>\varphi(z)$, where $\langle\varphi'(y^j),u^j\rangle=\max_{x\in D}\langle\varphi'(y^j),x\rangle$.

Proof. By the definition of u^j, we have

$$\max_{x\in D}\langle f'(y^j),x-y^j\rangle=\langle f'(y^j),u^j-y^j\rangle$$

since f is convex,

$$f(u)-f(v)\ge\langle f'(v),u-v\rangle$$

holds for all $u,v\in\mathbb{R}^n$. Therefore, the assumption in the lemma implies that

$$f(u^i)-f(z)=f(u^i)-f(y^i)\ge\langle f'(u^i),u^i-y^i\rangle>0.$$

Now we can construct an algorithm for solving problem (1) approximately.

Algorithm MAX

Step 1. Choose $x^k\in D, k:=0. z^k=argloc\max_{x\in D}\varphi(x)$, and m is given.

Step 2. Construct an approximation set $A_{z^k}^m$ at z^k

Step 3. Solve Linear programming problems:

$$\max_{x\in D}\langle\varphi'(y^i),x\rangle, \qquad i=1,2,\dots,m.$$

Let u^i be solutions to above problems:

$$\langle\varphi'(u^i),x\rangle,=\max_{x\in D}\langle\varphi'(y^i),x\rangle, \qquad i=1,2,\dots,m$$

Step 4. Compute η_k:

$$\eta_k=\max_{1\le i\le m}\langle\varphi'(y^i),u^i-y^i\rangle=\langle\varphi'(y^j),u^j-y^j\rangle.$$

Step 5. If $\eta_k>0$ then $x^{k+1}:=u^j$, $k:=k+1$ and go to Step 1.

Step 6. Terminate, z^k is an approximate global solution.

Lemma 4. *If $\eta_k>0$ for all $k=0,1,\dots$, then the sequence $\{z^k\}$ contructed by the Algorithm MAX is a relaxation sequence, i.e,*

$$f(z^{k+1})>f(z^k), \qquad k=0,1,\dots.$$

The proof follows from Lemma 3.

Results

We consider the problem of the following type:

$$\max_{x\in D}\left\{\varphi(x)=\frac{\langle Ax,x\rangle}{\langle Bx,x\rangle}\right\}$$

where $D=\{x\in\mathbb{R}^n \mid \alpha_i\le x_i\le\beta_i, i=1,2,\dots,n\}$.

The following problem has been solved numerically on PYTHON by the case the global solutions are found. Numerical experiment:

$$A=\begin{pmatrix}1 & 0 & \cdots & 0\\ \frac{1}{2} & \frac{1}{2} & \cdots & 0\\ \frac{1}{3} & \frac{1}{3} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n}\end{pmatrix}, B=\begin{pmatrix}-1 & -1 & \cdots & -1\\ -1 & -2 & \cdots & -1\\ \vdots & \vdots & \ddots & \vdots\\ -1 & -1 & \cdots & -n\end{pmatrix},$$

$$D=\{x\in\mathbb{R}^n \mid 1\le x_i\le 10, i=1,2,\dots,n\}.$$

Table 1. Numerical results

n	Conditional Gradient Algorithm		
	max	k	time(ms)
10	0.0689655	1	7.95
20	0.0338983	1	44.6
30	0.0224595	1	45.9
40	0.0167893	1	78.8
50	0.0133869	1	138
60	0.0111219	1	222

Conclusions

The purpose of this study was to find the maximum of a fractional programming problem with the convex numerator and concave denominator functions. So far, we have calculated the local maximum at the first step in the algorithm for finding the maximum using the conditional gradient method. The numerical experiment has been done by Python on up to 60 dimensions. Further, we aim to complete the algorithm for finding the maximum and plan to perform numerical experiments on up to 200 dimensions.

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