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Homogeneous Fractional Programming

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Homogeneous functions

In the 18th century, a homogeneity with respect to the uniform dilation (known also as the *standard homogeneity*) was studied by Leonhard Euler. Let us describe some properties of homogeneous functions (see e.g. [1]).

Definition A function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be standard homogeneous if there exists a number $k \in \mathbb{R}$ such that

$$f(\lambda x) = \lambda^k f(x), \forall \lambda > 0, \forall x \in \mathbb{R}^n.$$

The number k is called the *homogeneity degree* of the function f.

Corollary 1 For any standard homogeneous function with $k \neq 0, f(0) = 0$.

According to this definition, any linear function has the homogeneity degree 1, but the quadratic one ($f(x) = \langle x, Qx \rangle$, where $\langle .,. \rangle$ is the usual dot product) is homogeneous of the degree 2.

Euler's theorem A continuously differentiable function $f: \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree k if and only if $\langle \nabla f(x), x \rangle = k f(x), x \in \mathbb{R}^n$.

Corollary 2 For the homogeneity degree 0, $\langle \nabla f(x), x \rangle = 0$.

Sum-of-ratios programming

Consider the following sum-of-ratios function (see e.g. [2,3]):

$$\varphi(x) = \sum_{i=1}^{m} \frac{h_i(x)}{g_i(x)}$$

where numerators $h_i(x)$ and denominators $g_i(x)$ are standard homogeneous continuously differentiable functions with the same homogeneity degree $k \neq 0$. In this case $\varphi(\lambda x) = \varphi(x), \ \forall \lambda > 0$.

Searching for a solution of an optimization problem in this case, we do not consider points where the individual summands are not defined:

$$G_i = \{x: \ g_i(x) = 0\}, i = 1, \dots, m,$$

$$G_0 = \{x: \ x \in \cup_{i=1}^{m} G_i\}.$$

Let us formulate our problem as

$$\inf_{x \notin G_0} \varphi(x) \qquad \text{(I)}$$

Usually, in sum-of-ratios problems we suppose that $g_i(x) > 0$ and therefore G_0 is empty (see e.g. [3]). Here, it contains at least the origin.

Main result

If the minimum in (I) is achieved, the function has minimum value on a ray $\boldsymbol{r}$:

$$x_{opt} \in \operatorname{Arg} \min_{x \notin G_0} \varphi(x),$$
$$\boldsymbol{r} = \{x : x = \lambda x_{opt}, \lambda > 0\}.$$

Now, consider our optimization problem with an additional constraint set $\boldsymbol{C}$:

$$\inf_{x \notin G_0, \, x \in \boldsymbol{C}} \varphi(x) \qquad \text{(II)}$$

We also suppose that we can inscribe a ball with nonzero radius and center at **0** into $\boldsymbol{C}$. Then we have the following

Theorem If the minimum is achieved in both (I) and (II), then

$$\boldsymbol{r_{opt}} = \boldsymbol{r} \cap \boldsymbol{C} \in \operatorname{Arg} \min_{x \notin G_0, \, x \in \boldsymbol{C}} \varphi(x).$$

According to this theorem, if we have problem (II), we can first disregard constraints $\boldsymbol{C}$ and solve the problem (I) instead (if that problem is easier to solve). Having a solution for the problem (I), we can tailor it for the problem (II) using the constraint set.

Application to linear constraints

Let's now study a particular form of the constraint set – a polyhedron:

$$\boldsymbol{C} = \{x: \ \langle a_j, x \rangle \leq b_j, \ b_j > 0, j = 1, \dots, N\}.$$

Finding any x_{opt} for the first problem, we can deduce a solution for the second one as follows:

$$\lambda_j = b_j / \langle a_j, x \rangle \ \ j = 1, \dots, N,$$

$$\lambda_{opt} = \min_{\lambda_j > 0, \, j \in \{1, \dots, N\}} \lambda_j,$$

$$\boldsymbol{r_{opt}} = \{x : x = \lambda x_{opt}, 0 < \lambda \leq \lambda_{opt}\}.$$

Optimization on the unit sphere

Let us consider the simplest case with $G_0 = \{\boldsymbol{0}\}$. Since $\varphi(x)$ is constant along any ray in $\mathbb{R}^n$, it is enough to study optimization on the unit sphere, i.e. on the manifold defined by $\langle x, x \rangle = 1$.

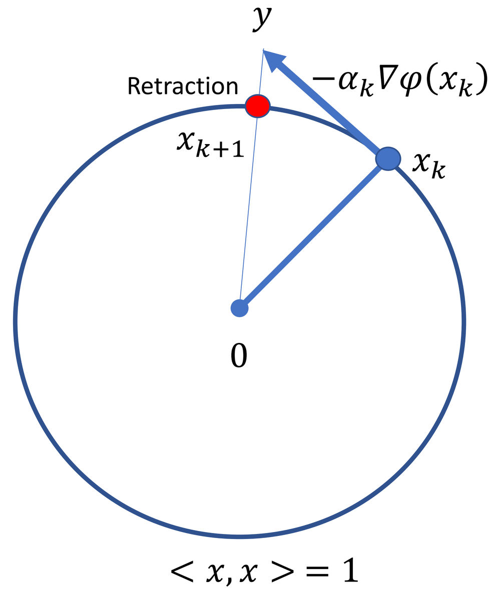
This idea is particularly useful, because some important convergence results for optimization methods fail for $\varphi(x)$ in general in its domain $\mathbb{R}^n \backslash G_0$ [4]. The unit sphere is an example of a *Riemannian manifold*. Optimization over Riemannian manifolds (see e.g. [4-6]) has drawn a lot of attention due to its applications in many fields, including low-rank matrix completion, phase retrieval, phase synchronization, blind deconvolution, and dictionary learning. To perform the optimization numerically, one can use e.g. the Manopt toolbox, available in Matlab, Python and Julia languages (www.manopt.org).

To illustrate basics of optimization on manifolds, consider a gradient step from the point x_k on the unit sphere (see the picture below). The gradient

$$\nabla \varphi(x) = \sum_{i=0}^{m} \left(\frac{1}{g_i(x)} \nabla h_i(x) - \frac{h_i(x)}{g_i^2(x)} \nabla g_i(x) \right)$$

is always lying in the tangent space (for any sphere point, $\langle \nabla \varphi(x), x \rangle = 0$). Suppose we make a step along direction $-\nabla \varphi(x_k)$. The resulting point is always outside the unit ball. We can retract the point back to the sphere:

$$y = x_k - \alpha_k \nabla \varphi(x_k), \alpha_k > 0,$$
$$x_{k+1} = \frac{1}{\|y\|_2} y.$$



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